Transformation Theory for the Adiabatic Compressible Turbulent Boundary Layer with Pressure Gradient

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We have reviewed Coles' analytical development of the transformation and its application to the constant pressure turbulent boundary layer. The limitations of the theory are defined, and an attempt is made to explain the discrepancies. The equations governing the scaling functions (which link the high-speed boundary layer to an equivalent low-speed flow) are derived in terms of the integral properties of the low-speed flow for an aribtrary pressure distribution. The modeling is completed by coupling the pressure gradients of the two flows and predicting the behavior of the integral properties of the low-speed flow. For a supersonic turbulent boundary layer with specified $p(x)$, the resulting formulation predicts the complete mapping of an equivalent low-speed flow. In addition, the integral properties of the high-speed flow are also predicted.

Nomenclature

A = Mach number parameter, $A = \{[(\gamma-1)/2]M_e^2\}\{1+[(\gamma-1)/2]M_e^2\}$
C_D = dissipation integral, $C_D = \int_0^\infty \frac{\tau}{\frac{1}{2}\rho_e u_e^2}\frac{\partial u/u_e}{\partial y}dy$
C_f = skin-friction coefficient, $C_f = \tau_w/\frac{1}{2}\rho_e u_e^2$
$\bar{f}$ = low-speed skin-friction parameter, $\bar{f} = (C_f/2)^{1/2}$
$\langle f\rangle,\langle f^2\rangle$ = Coles' sublayer constants, $\langle f\rangle = 17.2$, $\langle f^2\rangle = 305$
G = Clauser's equilibrium parameter (see Appendix)
H = form factor, $H = \theta/\delta^*$
J = viscous dissipation parameter, $J = \{(\bar{R}_{\bar{\theta}}/R_\theta)/(T_W/T_\infty)^3\}^{1/2}$
k = Von Kármán's constant, $k = 0.40$
M = Mach number
m = Mach number parameter, $m = [(\gamma-1)/2]M^2$
p = static pressure
R_L = Reynolds number, $R_L = \rho_\infty u_\infty L/\mu_\infty$
T = static temperature
T_0 = total temperature
T' = ratio of turbulent to laminar shear stress ($T' = \tau_t/\tau_l$)
u = velocity component parallel to freestream
u_τ = friction velocity, $u_\tau = (\tau_w/\rho_w)^{1/2}$
v = velocity component normal to freestream
x = coordinate parallel to freestream
y = coordinate normal to freestream
$\bar{\beta}_T$ = Clauser's pressure gradient parameter, $\bar{\beta}_T = (\bar{\delta}^*/\bar{\tau}_w)/d\bar{p}/d\bar{x}$
γ = ratio of specific heats, $\gamma = c_p/c_v$
δ = boundary-layer thickness
δ^* = displacement thickness, $\delta^* = \int_0^\infty\left(1-\frac{\rho u}{\rho_e u_e}\right)dy$
η,η' = scaling function
θ = momentum thickness, $\theta = \int_0^\infty\left(1-\frac{u}{u_e}\right)\frac{\rho u}{\rho_e u_e}dy$
μ = viscosity
ν = kinematic viscosity, ($\nu = \mu/\rho$)
ξ,ξ' = scaling function
$\tilde{\pi}$ = low-speed velocity profile parameter
ρ = density
σ,σ' = scaling function
τ = shear stress
ψ = stream function

Superscripts and subscripts

$(\ )_\infty$ = refers to upstream infinity
$(\ )_e$ = refers to boundary-layer edge
$(\ )_w$ = refers to wall condition
$(\bar{\ })$ = refers to low-speed flow
$(\ )_t$ = refers to turbulent flow
$(\ )_l$ = refers to laminar flow

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1. Introduction

THE dynamic equations governing the boundary layer are identical in form for laminar and turbulent flow with the exception of the specific form for the shear stress. Although the simple Newtonian stress relationship is valid everywhere for laminar flow, it is known to be valid only very near the wall in a turbulent boundary layer. The prediction of the turbulent boundary layer is, of course, much more difficult because of the lack of understanding of the turbulent momentum transport. Most analytical models rely on a simple gradient diffusion process (eddy viscosity); and while such a method is perhaps adequate for low-speed flow, the effect of compressibility on such a model is unknown. This difficulty, in addition to the existence of a vast amount of empirical knowledge about the low-speed turbulent boundary layer, makes the prospect of using a mathematical transformation to account for the effects of compressibility particularly inviting.

The objective of the present study is to critically examine the application of the transformation theory to turbulent boundary layers and to extend Coles' analysis to include pressure gradient.

We begin by reviewing Coles'[1] analytical development of the transformation and its application to the constant pressure turbulent boundary layer. The inaccuracies of the theory first noted by Baronti and Libby[2] are clarified, and an attempt is made to explain the remaining discrepancies.

The equations governing the scaling functions (which link the high-speed boundary layer to an equivalent low-speed flow) are derived in terms of the integral properties of the low-speed flow for an arbitrary pressure distribution. The modeling is completed by relating the pressure gradients of the two flows and predicting the behavior of the integral properties of the low-speed flow.

Because of the lack of an experimental modeling of a high-speed and a low-speed boundary-layer flow, a critical evaluation of the theory is not possible for the variable pressure case at this time. However, in order to check the over-all soundness of the formulation, comparisons are made with the integral properties of available high-speed experiments.

2. Review of Coles' Transformation

The use of a mathematical transformation to account for the effect of compressibility has been extensively applied to laminar boundary layers and wakes. Howarth,[3] Stewartson,[4] and many others have used this technique. Coles has formulated perhaps the most general transformation yet considered for the dynamic boundary layer equations. These equations for the steady, two-dimensional mean flow of a compressible fluid are

$$(\partial/\partial x)(\rho u) + (\partial/\partial y)(\rho v) = 0 \qquad (1)$$

$$\rho u \partial u/\partial x + \rho v \partial u/\partial y = -dp/dx + \partial \tau/\partial y \qquad (2)$$

The transformation,§ simply stated, specifies the correspondence between a flow governed by these equations and an incompressible flow for which

$$\partial \bar{u}/\partial \bar{x} + \partial \bar{v}/\partial \bar{y} = 0 \qquad (3)$$

$$\bar{\rho}\bar{u}\partial \bar{u}/\partial \bar{x} + \bar{\rho}\bar{v}\partial \bar{u}/\partial \bar{y} = -d\bar{p}/d\bar{x} + \partial \bar{\tau}/d\bar{y} \qquad (4)$$

and $\bar{\rho}$ is constant.

In the development of the transformation by Coles, he first argues for three independent scaling functions, σ, ξ, and η defined by

$$\bar{\psi}/\psi = \sigma \qquad (5)$$

$$d\bar{x}/dx = \xi \qquad (6)$$

$$(\bar{\rho}/\rho)\partial \bar{y}/\partial y = \eta \qquad (7)$$

In order to insure the boundedness of the pressure gradient throughout each flowfield, Coles imposes the condition that these functions be independent of y, i.e., $\sigma = \sigma(x)$, $\eta = \eta(x)$ and $\xi = \xi(x)$. Even with this restriction, the transformation remains very general. It is possible, as will be discussed below, to draw some conclusions about the behavior of the compressible flow *without any further development of the theory*, i.e., by just using Eqs. (5–7). If any additional development of the theory is pursued, these conclusions must still hold if the transformation is to be valid at all. One such conclusion of particular importance is the "law of corresponding stations"

$$\bar{C}_f \bar{R}_{\bar{\theta}} = (\rho_e \mu_e / \rho_w \mu_w) C_f R_\theta \qquad (8)$$

This relationship can be derived by simply noting that Newtonian friction must hold at the wall. (The derivation, which is quite straightforward, can be found in Ref. 1.) A violation of this law represents a violation of either: 1) the form of the transformation represented by $\sigma(x)$, $\eta(x)$, $\xi(x)$, or 2) Newtonian friction at the wall.

The importance of the law, as its name implies, is that it defines the correspondence of stations in the compressible and the equivalent incompressible flow. What is meant by equivalence is that the histories of the two flows must correspond in some meaningful way. For example, previous investigators have argued on physical grounds that constant pressure in the compressible flow must require constant pressure in the low-speed flow. Hence, if one can neglect initial conditions, the supersonic and incompressible flat plate boundary layers should be equivalent. Further, this physically realistic argument has been substantiated mathematically (see Sec. 3), and hence the constant pressure boundary layer provides a straightforward application of the transformation.

Before we pursue the extension of the transformation theory to variable pressure, it is appropriate to comment on the ability of the transformation to predict the behavior of compressible boundary layers at constant pressure.

There are two predictions which are of equal importance but which can, and should, be considered separately. First, while Eq. (8) specifies the corresponding stations, it does not provide the needed relationship between C_f and $\bar{C}_f$. For this we must resort to hypothesis. Coles has argued the plausibility of the existence of a "turbulent substructure" which is essentially an argument for the invariance with compressibility of the sublayer Reynolds number. For constant pressure, adiabatic flows, his correlation is excellent, and in the following sections we shall assume that this approach is valid even for variable pressure. The second prediction involves the velocity field.

We observe that it is possible to compare the velocity profiles in the coordinates

$$\frac{u}{u_e} = \frac{\bar{u}}{\bar{u}_e} \quad \text{vs} \quad \frac{\bar{y}}{\bar{\theta}} = \int_0^y \frac{\rho}{\rho_e}\frac{dy}{\theta}$$

and using Eq. (8) to define the corresponding stations, this comparison can be carried out without further development of the theory, for example, without need of a skin-friction law, C_f vs $\bar{C}_f$. This is because the coordinates chosen eliminate the need to specify any of the scaling functions, and importantly we do not need to resort to using the detailed shape of the velocity profiles to define the transformation which we are trying to check. (The approach of choosing a value of $\bar{C}_f$ which allows the best agreement of the velocity profiles in the inner logarithmic region of the boundary-layer precludes any test of the transformation in this region since it forces a match there.)

The result of applying the transformation in this way to the experimental data of Coles[5] and Korkegi[6] (adiabatic, flat plate) is shown in Fig. 1, contrasted with the low-speed data of Wieghardt[7] and the analytical form suggested by Coles.[1] As can be seen, there is a deviation from the low-speed profiles which increases as Mach number increases. The deviation is most significant in the inner region of the boundary layer. In addition, the thickness of the boundary layer is increased. The actual magnitude of the velocity in the wake region is within 2–3% of the predicted value whereas the deviation at $\bar{y}/\bar{\theta} = 1$ is seen to be as large as 10%.¶

Apparently, then, the transformation fails in the inner "law of the wall" region of the boundary layer to an extent which increases systematically with Mach number. Consequently, the fundamental idea of Coles that a simple geometric mapping of a low-speed to a high-speed flow can account for effects of compressibility, does not appear to be valid in detail.

An important process affecting turbulent momentum exchange in the region adjacent to the laminar sublayer is the "damping" of convected velocity fluctuations by viscous forces there. For an adiabatic supersonic boundary layer, the temperature near the wall is significantly higher than ambient, and an increased dissipation of turbulent velocity fluctuations due to the increased viscosity is therefore expected in this region. This increase in viscous dissipation is believed to be a plausible explanation for the deviation of the transformed profiles in the law of the wall region. To give some quantitative support to this conjecture, we have made a rough estimate of this effect in the following way. Webb[9] has arrived at an empirical estimate of the effect of dissipation based on a simple gradient diffusion mixing length model

§ The term transformation will be used here in the same sense used by Coles, i.e., as a true mapping of one flowfield into another rather than simply a mathematical manipulation.

¶ A similar conclusion has also been reached recently by Green.[8]

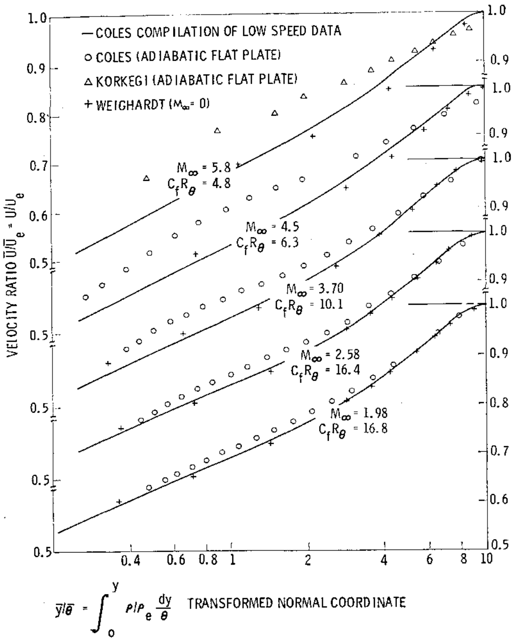


Fig. 1 Transformed high-speed velocity profiles.

which seems to give an accurate representation of the turbulent shear stress near the wall in confined low-speed turbulent flows. This estimate is obtained by applying the known solution for the decay of velocity in a finite viscous vortex to the estimate of the damping of the turbulent velocity fluctuations which are convected during the mixing process. The result is that the local shear stress is reduced by a factor dependent on the local ratio of turbulent to laminar shear stress, $T' = \tau_t/\tau_l$.

A typical value of the ratio T', evaluated at $y = \theta$, is estimated for a compressible boundary layer as follows:

$$T' \sim \rho\theta^2(\partial u/\partial y)^2/(\mu\partial u/\partial y) \sim (\theta^2/\nu_w)(u_\tau/\theta)$$

$$\sim (\theta/\nu_w)(\tau_w/\rho_w)^{1/2} \sim R_\theta C_f^{1/2}(\mu_e/\mu_w)\left(\frac{\rho_w}{\rho_e}\right)^{1/2}$$

Taking $\rho \sim T^{-1}$ and $\mu \sim T$ and recalling that $C_f R_\theta = \bar{C}_f \bar{R}_\theta$, then the ratio of T' for low-speed flow to its value for compressible flow is

$$\mathfrak{J} = \overline{T'}/\overline{T'} = \{(\bar{R}_\theta/R_\theta)(T_w/T_e)^3\}^{1/2}$$

To see whether the parameter $\mathfrak{J}$ indeed collapses the velocity profiles for various Mach numbers and Reynolds numbers, data taken by Coles[5] and by Lobb, Winkler and Persch[10] over a range of M_∞ up to 8.2 (including heat transfer) for $\mathfrak{J} \simeq 6$ are shown in Fig. 2.

On the basis of this correlation,** we tentatively conclude that viscous dissipation effects are responsible for the deviation of the transformation theory near the wall and, at least in principle, the transformation theory is invalid. However, for not excessively high wall temperature, the deviation is small; it extends over only a small fraction of the boundary layer, and it does not preclude the application of the theory over most of the layer. An interesting extension of the theory would be to take into account this dissipation effect near the wall; however, we have made no attempts to do so.

3. Generalization to Variable Pressure (Adiabatic Wall)

For the constant pressure turbulent boundary layer, Coles derives the following relationships for the scaling functions σ, η, ξ:

$$\sigma/\eta = (\bar{u}_\infty/u_\infty)(\text{const}) \tag{8}$$

$$\xi/\eta = (\rho_w\mu_w/\bar{\rho}\bar{\mu})d(\sigma\theta)/d\theta \tag{9}$$

$$\frac{\bar{\mu}}{\sigma\mu_\infty} = \frac{T_w}{T_\infty} - \langle f\rangle \frac{T_w - T_{0\infty}}{T_\infty}(\bar{C}_f/2)^{1/2} - \langle f^2\rangle m_\infty \frac{\bar{C}_f}{2} \tag{10}$$

$$(\mu/T = \text{const})$$

He arrives at these equations by invoking the condition of constant pressure, Newtonian friction at the wall, and an assumption known as the "substructure hypothesis." The analysis is well detailed in Coles' report and will not be repeated here.

If we take the substructure hypothesis to be independent of pressure gradient, we need only to generalize Eqs. (8) and (9). This assumption has been made in the following developments. Although plausible, it is experimentally unverified.

The pressure gradients of the two flows are related[1] by

$$\frac{d\bar{p}}{d\bar{x}} = \frac{\bar{\rho}\sigma^2}{\xi\eta^2}\left(\frac{1}{\rho_e}\frac{dp}{dx} + \frac{u_e^2}{\eta/\sigma}\frac{d(\eta/\sigma)}{dx}\right) \tag{11}$$

This equation is one of the required relationships; however, it is in a form which is not useful for our purposes in that it involves both $d\bar{p}/d\bar{x}$ and dp/dx, one of which is unknown a priori. However, it can be shown that by noting the behavior of the velocity in the vicinity of the wall[12] the pressure gradients are linked by the following relationship:

$$(\rho_e/\rho_w)(\theta/\tau_w)dp/dx = (\bar{\theta}/\bar{\tau}_w)d\bar{p}/d\bar{x} \tag{12}$$

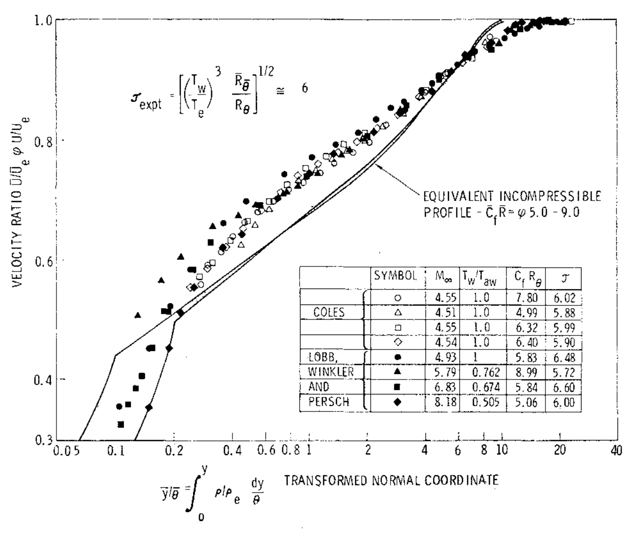
Combining Eqs. (11) and (12) after some manipulation, we find that

$$\frac{1}{\eta/\sigma}\frac{d\eta/\sigma}{d\bar{x}} = \left\{1 - \frac{\rho_w}{\rho_e}\frac{\rho_w\mu_w}{\bar{\rho}\bar{\mu}}\frac{\sigma\eta}{\xi}\right\}\frac{1}{\bar{\rho}\bar{u}_e^2}\frac{d\bar{p}}{d\bar{x}} \tag{13}$$

where σ, η, ξ are taken as functions of $\bar{x}$ rather than x. Equation (13) replaces the constant pressure condition $\eta/\sigma = (u_\infty/\bar{u}_\infty)(\text{const})$.

One more equation is required, and for this we utilize the relationship between the shear stress terms of the momentum equation,[1]

$$\frac{\partial\bar{\tau}}{\partial\bar{y}} = \frac{\bar{\rho}\sigma^2}{\xi\eta^2}\left[\frac{1}{\rho}\left(\frac{\partial\tau}{\partial y} - \frac{\psi}{\sigma}\frac{\partial u}{\partial y}\frac{d\sigma}{dx}\right) + \frac{dp}{dx}\left(\frac{1}{\rho_e} - \frac{1}{\rho}\right) + \frac{1}{\eta/\sigma}\frac{d\eta/\sigma}{dx}(u_e^2 - u^2)\right] \tag{14}$$



	SYMBOL	M_∞	T_w/T_{aw}	$C_f R_\theta$	$\mathcal{J}$
COLES	○	4.55	1.0	7.80	6.02
	△	4.51	1.0	4.99	5.88
	□	4.55	1.0	6.32	5.99
	◇	4.54	1.0	6.40	5.90
LOBB, WINKLER AND PERSCH	●	4.93	1	5.83	6.48
	▲	5.79	0.762	8.99	5.72
	■	6.83	0.674	5.84	6.60
	◆	8.18	0.505	5.06	6.00

Fig. 2 High-speed velocity profiles for J = const.

** It has also been shown that the deviation of the profiles can be plotted as a function of J and a good estimate of the deviation can be made using the analysis of Ref. 9 (also see Ref. 11).

Integrating across the boundary layer, we find that

$$\frac{\xi\eta}{\sigma^2}\bar{\tau}_w = \tau_w + \frac{1}{\sigma}\frac{d\sigma}{dx}\cdot\int_0^\delta \psi\frac{\partial u}{\partial y}dy + \frac{dp}{dx}\cdot\int_0^\delta(1-\rho/\rho_e)dy - \frac{1}{\eta/\sigma}\frac{d\eta/\sigma}{dx}\cdot\rho_e u_e^2\cdot\int_0^\delta\left(1-\frac{u^2}{u_e^2}\right)\frac{\rho}{\rho_e}dy$$

Noting that

$$\int_0^\delta\left(1-\frac{u^2}{u_e^2}\right)\frac{\rho}{\rho_e}dy = \frac{\bar{\rho}}{\rho_e\eta}(\bar{\delta}^* + \bar{\theta})$$

and evaluating the integral

$$\int_0^\delta(1-\rho/\rho_e)dy$$

by approximating the temperature field†† by the equation $(T_0/T_w) - 1 \simeq (u/u_e)[(T_{0\infty}/T_w) - 1]$ we arrive at the required relationship

$$\frac{\rho_w\mu_w}{\bar{\rho}\bar{\mu}}\frac{\sigma\eta}{\xi} = \frac{\bar{C}_f/2 - (\bar{\theta}/\sigma)d\sigma/d\bar{x} + (\bar{\delta}^*/\bar{\rho}\bar{u}_e^2)(d\bar{p}/d\bar{x})(1+\bar{H})}{\bar{C}_f/2 + [1 + (T_{0\infty}/T_w)\bar{H}](\bar{\delta}^*/\bar{\rho}\bar{u}_e^2)d\bar{p}/d\bar{x}} \tag{15}$$

We have for convenience redefined the scaling functions in the following manner:

$$\sigma = \frac{\bar{\mu}}{\mu_\infty}\sigma',\quad \xi = [(\rho_\infty u_\infty/\mu_\infty)/(\bar{\rho}\bar{u}_\infty/\bar{\mu})]\xi',\quad \eta/\sigma = (u_\infty/\bar{u}_\infty)\eta'/\sigma' \tag{16}$$

The resulting equations become

$$\frac{1}{\sigma'} = (1+m_\infty)\left[\frac{T_w}{T_{0\infty}} - \langle f\rangle\left(\frac{T_w}{T_{0\infty}} - 1\right)\bar{f} - \langle f^2\rangle\frac{m_e}{1+m_e}\bar{f}^2\right] \tag{17}$$

$$\frac{1}{\eta'/\sigma'}\frac{d\eta'/\sigma'}{d\bar{R}_{\bar{x}}} = \left\{\frac{T_{0\infty}}{T_w}(1+m_e)^{-1}\frac{p_e}{p_\infty}\frac{\sigma'\eta'}{\xi'} - 1\right\}\frac{1}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} \tag{18}$$

$$\frac{p_e}{p_\infty}\frac{\sigma'\eta'}{\xi'} = \frac{\bar{f}^2 - (\bar{R}_{\bar{\theta}}/\sigma')d\sigma'/d\bar{R}_{\bar{x}} - (1+\bar{H})(\bar{R}_{\bar{\delta}^*}/\bar{u}_e)d\bar{u}_e/d\bar{R}_{\bar{x}}}{\bar{f}^2 - [1 + (T_{0\infty}/T_w)\bar{H}](\bar{R}_{\bar{\delta}^*}/\bar{u}_e)d\bar{u}_e/d\bar{R}_{\bar{x}}} \tag{19}$$

where

$$\frac{p_e}{p_\infty} = \left\{1 - m_\infty\left(\frac{u_e^2}{u_\infty^2} - 1\right)\right\}^{\gamma/(\gamma-1)},\quad \frac{u_e}{u_\infty} = (\eta'\sigma')\frac{\bar{u}_e}{\bar{u}_\infty},\quad \bar{f} = \left(\frac{\bar{C}_f}{2}\right)^{1/2}$$

and

$$m_e = \frac{A}{1-A},\quad A = \frac{m_\infty}{1+m_\infty}\left(\frac{\eta'}{\sigma'}\right)^2\left(\frac{\bar{u}_e}{\bar{u}_\infty}\right)^2 = \frac{m_\infty}{1+m_\infty}\left(\frac{u_e}{u_\infty}\right)^2$$

Hence, given a low-speed flow which is completely specified (at least insofar as its integral properties are concerned), these three relationships can be used to construct an equivalent supersonic flow for a given M_∞.

The objective of the study is, of course, to take a specified velocity distribution in the supersonic flow $u_e(R_x)$ and predict the necessary $\bar{u}_e(\bar{R}_{\bar{x}})$ to produce its low-speed equivalent.

†† Since only adiabatic flows are being considered, the variation of the total temperature is small and the approximation used should be sufficiently accurate.

Returning to Eq. (12), it can be shown that

$$\frac{1}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} = \frac{T_w}{T_{0\infty}}(1+m_e)\left(\frac{p_e}{p_\infty}\right)^{-1}\frac{1}{\sigma'\eta'}\frac{1}{u_e}\frac{du_e}{dR_x} \tag{20}$$

Hence, given $u_e(R_x)$, we have an equation to make such a prediction provided that σ' and η' are known. These functions can be predicted by simply calculating the integral properties of the low-speed flow simultaneously. Coupling such a calculation with the previously derived transformation laws, we have then arrived at a self-contained set of equations which predicts the modeling we seek. The integral properties of the supersonic turbulent boundary layer are also obtained.

Returning to the scaling functions, we note that Eq. (19) requires $d\sigma'/d\bar{R}_{\bar{x}}$, and hence, for simplicity, we have treated σ' as one of the dependent variables of the final system of ordinary differential equations. We differentiate Eq. (17), which becomes

$$\frac{2\langle f^2\rangle A\bar{f}^2}{\eta'}\frac{d\eta'}{d\bar{R}_{\bar{x}}} - \left[\frac{1}{\sigma'^2}\frac{1}{(1+m_\infty)} + \frac{2\langle f^2\rangle A\bar{f}^2}{\sigma'}\right]\frac{d\sigma'}{d\bar{R}_{\bar{x}}} + \left[\langle f\rangle\left(\frac{T_w}{T_{0\infty}} - 1\right) + 2\langle f^2\rangle A\bar{f}\right]\frac{d\bar{f}}{d\bar{R}_{\bar{x}}} + 2A\langle f^2\rangle\bar{f}^2\frac{1}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} = 0 \tag{21}$$

Combining Eqs. (18) and (19)

$$\left[\bar{f}^2 - \left(1 + \frac{T_{0\infty}}{T_w}\bar{H}\right)\frac{\bar{R}_{\bar{\delta}^*}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}}\right]\frac{1}{\eta'}\frac{d\eta'}{d\bar{R}_{\bar{x}}} - \left[\bar{f}^2 - \left(1 + \frac{T_{0\infty}}{T_w}\bar{H}\cdot\frac{2+m_e}{1+m_e}\right)\frac{\bar{R}_{\bar{\delta}^*}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}}\right]\frac{1}{\sigma'}\frac{d\sigma'}{d\bar{R}_{\bar{x}}} = \left\{\left[\frac{T_{0\infty}}{T_w}\frac{1}{(1+m_e)} - 1\right]\bar{f}^2 + \left[1 - \frac{T_{0\infty}}{T_w}\frac{1}{1+m_e} + \frac{T_{0\infty}}{T_w}\bar{H}\frac{m_e}{1+m_e}\right]\frac{\bar{R}_{\bar{\delta}^*}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}}\right\}\frac{1}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} \tag{22}$$

and recalling $dR_x/d\bar{R}_{\bar{x}} = 1/\xi'$, it follows from Eq. (19) that

$$\frac{dR_x}{d\bar{R}_{\bar{x}}} = \left(\frac{p_e}{p_\infty}\sigma'\eta'\right)^{-1}\left[\frac{\bar{f}^2 - \bar{H}(\bar{R}_{\bar{\delta}^*}/\sigma')\dfrac{d\sigma'}{d\bar{R}_{\bar{x}}} - (1+\bar{H})(\bar{R}_{\bar{\delta}^*}/\bar{u}_e)\dfrac{d\bar{u}_e}{d\bar{R}_{\bar{x}}}}{\bar{f}^2 - (1 + (T_{0\infty}/T_w)\bar{H})(\bar{R}_{\bar{\delta}^*}/\bar{u}_e)d\bar{u}_e/d\bar{R}_{\bar{x}}}\right] \tag{23}$$

Equations (20–23), coupled with the equations which govern the low-speed boundary-layer flow, represent the total formulation. Since only the integral properties of the low-speed flow are required, we have for the present used an integral formulation for the low-speed flow (see Appendix). The complete formulation is composed of seven ordinary differential equations with the dependent variables: $\bar{R}_{\delta^*}$, $\tilde{\pi}$, $\bar{f}$ $\bar{u}_e/\bar{u}_\infty$, σ', η', R_x with $\bar{R}_{\bar{x}}$ as the independent variable. These equations are well defined provided $u_e(R_x)$ is specified, and the variables are initialized.

An important capability of the present analysis is a flexibility which allows for initialization at any station in the flow whether it is in a state of equilibrium or not. Theories which have an explicit coupling between C_f and R_θ do not have this capability. To perform the initialization, for a boundary layer at a given station R_x for which C_f, R_θ, H, M_e are defined and $T_w/T_{0\infty} = 1$,

$$\bar{f} = (C_{fe}/2)^{1/2}\left[\frac{(1+m_e)}{(1+\langle f^2\rangle m_e C_{fe}/2)}\right]^{1/2} \tag{24a}$$

$$\bar{H} = (1+m_e)H/(1-m_eH) \tag{24b}$$

$$\tilde{\pi} = \tilde{\pi}(\bar{H},\bar{f})\text{ (see Appendix)} \tag{24c}$$

$$1/\sigma' = (1+m_\infty)\{1 - \langle f^2\rangle[m_e/(1+m_e)]\bar{f}^2\} \tag{24d}$$

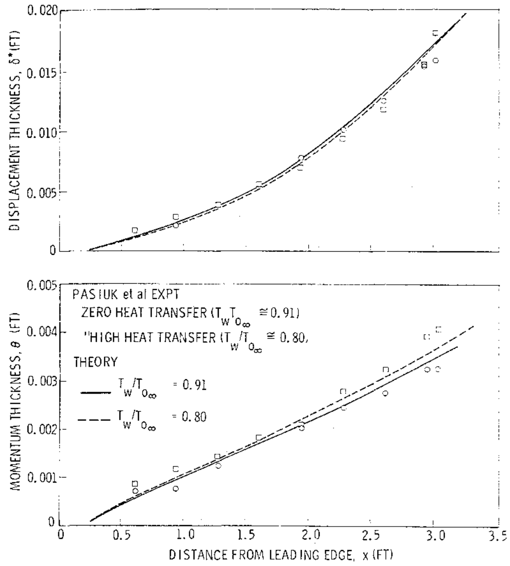


Fig. 3 Prediction of integral properties—two-dimensional expansion.

$$\bar{R}_{\bar{\delta}_\infty}{}^* = (p_e/p_\infty)\sigma'(M_e/M_\infty).$$

$$\{(1 - m_eH)/[(1 + m_\infty)(1 + m_e)]^{1/2}\} R_{\theta\infty}/H \quad (24e)$$

$$\eta' = \sigma'(M_e/M_\infty)[(1 + m_\infty)/(1 + m_e)]^{1/2} \quad (24f)$$

$\bar{u}_e/\bar{u}_\infty$ and $\bar{R}_{\bar{x}}$ are both arbitrary; hence, we can take $\bar{u}_e/\bar{u}_\infty = 1$ and $\bar{R}_{\bar{x}} = 0$.

The wall temperature $T_w/T_{0\infty}$ has been carried throughout the analysis, and, hence, formally nonadiabatic boundary layers could also be predicted. In a strict sense, however, Crocco's[13] suggestion that the heat transfer of the two flows must match can be shown to be a requirement of the mapping (see Ref. 12). In particular we must have

$$(\theta/T_e)(\partial T/\partial y)_w = (\bar{\theta}/\bar{T}_e)(\partial \bar{T}/\partial \bar{y})_w$$

and hence the low speed flow, while incompressible, cannot be of constant density if it is to precisely correspond to a nonadiabatic compressible flow.

4. Application of Theory

The objective of the study has not been to develop a theory solely capable of predicting the integral properties of a compressible turbulent boundary layer. We have sought to develop a mathematical transformation which would predict the effects of compressibility on the detailed properties (e.g., velocity distribution) of the turbulent boundary layer.

However, in the absence of an actual experimental modeling of a high-speed and a low-speed boundary-layer flow, which is required for a point-by-point checking of the theory, we have made comparisons with the integral properties of several high-speed experiments in order to check the over-all applicability of the formulation.

Two-Dimensional Expansion

Pasiuk, Hastings and Chatham[14] have measured the development of the turbulent boundary layer formed on a flat plate mounted in the diverging section of a supersonic nozzle. The freestream Mach number varied from 1 at the leading edge of the plate to approximately 3 at the farthest downstream station ($X = 3$ ft). No skin friction measurements were made and in order to start the calculation the following procedure was adopted. We have chosen the values noted by Coles[1] for which the "wake" of the incompressible boundary layer disappears as being characteristic of the low-speed turbulent boundary layer near its origin

$$\tilde{\pi}_0 = 0,\ \bar{C}_{f0} = 5.9 \times 10^{-3},\ x_0 = 0.25 \text{ ft}$$

$$\bar{R}_{\bar{\theta}0} = 4.25 \times 10^2 \text{ (assumed location of transition)}$$

The results of the calculations are shown in Fig. 3 where theory and experiment are seen to be in good agreement. The "heat transfer" case is really very nearly adiabatic, $T_w/T_{0\infty} = 0.80$ (cf adiabatic wall $T_w/T_{0\infty} = 0.91$), and hence both values of $T_w/T_{0\infty}$ were used in the calculations.

Two-Dimensional Compression

McLafferty and Barber[15] have investigated the behavior of a turbulent boundary layer in an adverse pressure gradient produced with a curved surface. The edge Mach number varied from 3 to approximately 1.9. Again no skin-friction measurements were made, and hence a flat plate calculation (starting at the origin with the initial conditions previously described) up to the experimental value of R_θ was made to determine C_{f0}. The calculations shown in Fig. 4 were made with $T_w/T_{0\infty} = 1$, for the theoretical (Prandtl-Meyer) Mach number distribution calculated for the experimental geometry, and with the flat plate C_{f0}. The experimental data (θ,δ^*) and the theoretical predictions are in good agreement up to approximately $S = 2$ in. After this, however, the theory predicts separation which is not observed experimentally. On the other hand, the theoretical calculation for $R_{\theta 0} = 4200$ shows separation beyond $S = 2$ in. whereas there was experimental evidence of separation near $S = 1.5$ in. A reduction‡‡ of C_{f0} by $\frac{1}{2}$ for this case moved the separation forward but less than $\Delta S = 0.5$ in. Nevertheless, for the high R_θ cases, the initial skin friction could account for most of the discrepancy. For $R_\theta = 2540$, various $M_e(x)$ and C_{f0} were tried, all with little effect on the result. Hence, we must conclude that the theory breaks down in this situation. Whether the fault lies in some unaccounted effect (e.g., there were significant experimental lateral pressure gradients) or in the low-speed theory or in the transformation is not known.

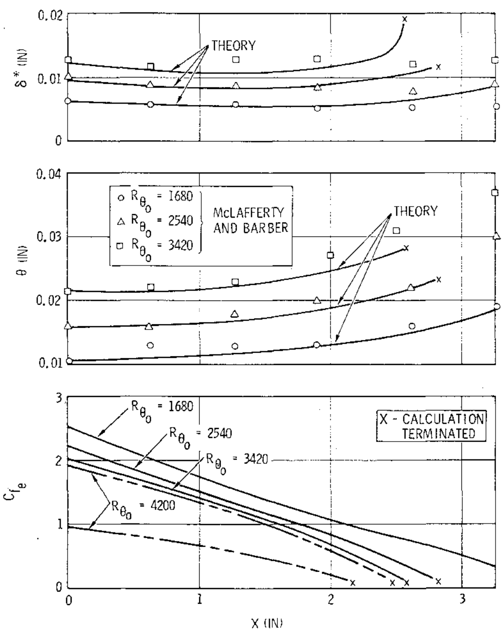


Fig. 4 Prediction of integral properties—two-dimensional compression.

‡‡ The boundary layer was thickened by air injection within 10δ of the curved ramp and hence the actual C_{f0} could well be less than the flat plate value.

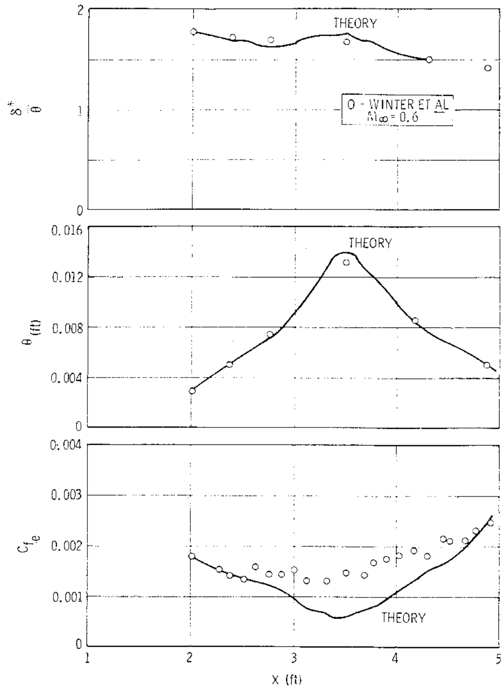


Fig. 5a Prediction of integral properties—axisymmetric waisted body at $M_\infty = 0.6$.

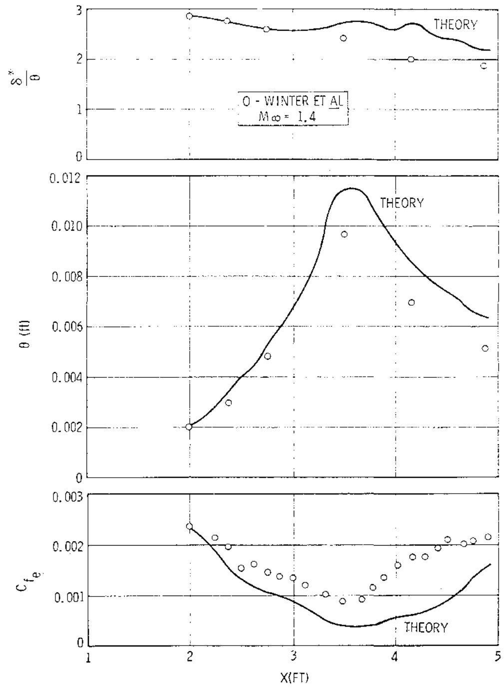


Fig. 5b Prediction of integral properties—axisymmetric waisted body at $M_\infty = 1.4$.

Axisymmetric Compression

Winter, Smith and Rotta[16] have made a complete set of measurements (including skin friction) of the turbulent boundary layer formed on a wasted body of revolution. While the theory is formulated for a two-dimensional flow, one can at least formally§§ utilize the Mangler transformation

$$x_{2D} = \int_0^{x_{\text{axi}}} \left(\frac{r_w}{r_{w0}}\right)^2 dx_{\text{axi}},\ y_{2D} = \frac{r_w}{r_{w0}}\, y_{\text{axi}},\ \tau_{2D} = \frac{\tau_{\text{axi}}}{r_w/r_{w0}}$$

§§ If there exists such a transformation, one can show that it must have this form by invoking the integral momentum equation and Newtonian friction at the wall.

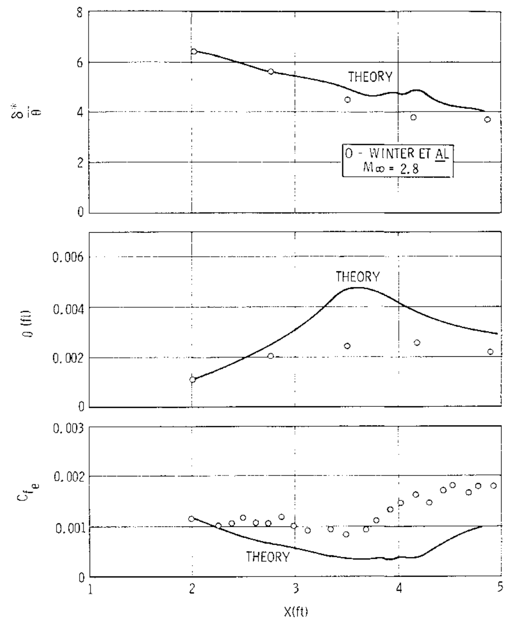


Fig. 5c Prediction of integral properties—axisymmetric waisted body at $M_\infty = 2.8$.

Figures 5a, b, c show the experimental results contrasted with the theoretical predictions. The form factor H is uniformly well predicted. The local skin-friction coefficient is typically twice the theoretical prediction near the waist of the body ($X = 3.5$ ft). The momentum thickness is fairly well predicted for $M_\infty = 0.6$ and 1.4, but the agreement at $M_\infty = 2.8$ is poor, again near the waist of the body.

There are experimental difficulties in this region (the integral momentum equation is not satisfied) and these may account for some of the disagreement. It is difficult to imagine their resulting in a factor of 2 error in the skin friction; hence, it must be assumed that the theory is in fault. Again it is not possible to pinpoint the origin of the problem however, since the discrepancy does not increase markedly with Mach number, we are inclined to rule out the compressibility transformation, leaving the low-speed formulation and the Mangler transformation in question.

Separation

We find that in the limit as $\bar{f} \to 0$, the low-speed formulation predicts that $(\bar{\theta}/\bar{u}_e)(d\bar{u}_e/d\bar{x}) = -4.05 \times 10^{-3}$.

By recalling Eq. (12) and noting that

$$\bar{C}_f/C_f = R_\theta/\bar{R}_{\bar{\theta}} = 1/(\sigma' T_e/T) \to T_w/T_\infty$$

we find that, independent of the history of the turbulent boundary layer,

$$\{(T_w/T_e)^2\ (\theta/\rho_e u_e^2)\ dp/dx\}_{\text{separation}} = 4.05 \times 10^{-3}$$

The result is in good agreement with the empirical correlation of Zukoski[17] which represents a compilation of adiabatic wall, turbulent boundary layer separation data for $M_\infty = 2 \to 6$.

5. Conclusions

In conclusion, we have found that the extension of the theory of Coles to variable pressure provides a useful general theory for compressible adiabatic boundary layers, a theory which requires no additional empirical factors beyond those needed to describe the low-speed boundary layer.

We have found that, in constant pressure boundary layers, velocity profiles depart in a systematic way with increasing wall temperature from the form predicted by the existence of the scaling functions $\sigma(x)$, $\eta(x)$, and $\xi(x)$. This departure occurs in the inner region of the boundary layer, and we have given arguments attributing it to viscous effects on the turbulent transport near the wall. We have concluded that this should not affect the application of the theory under a wide range of conditions.

When the theory is compared to available experiments on variable pressure layers, firm conclusions even to its general validity cannot be drawn owing to various deficiencies in the data obtained in these experiments. A definitive test of the transformation requires an experimental modeling of a high-speed and a low-speed flow, and we strongly recommend that such an experiment be performed.

Appendix

The formulation[18] used consists of the integrated form of the momentum equation

$$\bar{H}\frac{d\bar{R}_{\bar{\delta}^*}}{d\bar{R}_{\bar{x}}} + \bar{R}_{\delta^*}\frac{\partial \bar{H}}{\partial \bar{\pi}}\frac{d\bar{\pi}}{d\bar{R}_{\bar{x}}} + \bar{R}_{\bar{\delta}^*}\frac{\partial \bar{H}}{\partial \bar{f}}\frac{d\bar{f}}{d\bar{R}_{\bar{x}}} = \bar{f}^2 - (2\bar{H}+1)\frac{\bar{R}_{\bar{\delta}^*}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} \quad \text{(A1)}$$

and its first moment

$$\bar{J}\frac{d\bar{R}_{\bar{\delta}^*}}{d\bar{R}_{\bar{x}}} + \bar{R}_{\bar{\delta}^*}\frac{\partial \bar{J}}{\partial \bar{\pi}}\frac{d\bar{\pi}}{d\bar{R}_{\bar{x}}} + \bar{R}_{\bar{\delta}^*}\frac{\partial \bar{J}}{\partial \bar{f}}\frac{d\bar{f}}{d\bar{R}_{\bar{x}}} = \bar{C}_D - 3\bar{J}\frac{\bar{R}_{\bar{\delta}^*}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} \quad \text{(A2)}$$

and the differential form of the skin-friction law

$$\frac{d\bar{R}_{\bar{\delta}^*}}{d\bar{R}_{\bar{x}}} + \bar{R}_{\bar{\delta}^*}[P]\frac{d\bar{\pi}}{d\bar{R}_{\bar{x}}} + \bar{R}_{\bar{\delta}^*}[Q]\frac{d\bar{f}}{d\bar{R}_{\bar{x}}} = -\frac{\bar{R}_{\bar{\delta}^*}}{\bar{u}_e}\frac{d\bar{u}_e}{d\bar{R}_{\bar{x}}} \quad \text{(A3)}$$

where $\bar{\pi}$ comes from Coles[1] analytical form of the turbulent boundary layer,

$$\bar{u}/\bar{u}_\tau = (1/k)\ln(\bar{y}\bar{u}_\tau/\bar{\nu}) + C + (\bar{\pi}/k)W(\bar{y}/\bar{\delta})$$

$$k = 0.40 \text{ (von Kármán's constant)}$$

$C = 5.10$ and $\bar{u}_\tau = (\bar{\tau}_w/\bar{\rho})^{1/2}$ and $\bar{H}$, $\bar{J}$, $[P]$, $[Q]$, and $\bar{C}_D$ are specified functions of $\bar{\pi}$ and $\bar{f}$ (see Ref. 18).

In order to arrive at the dependence of $\bar{C}_D$ on $\bar{\pi}$ and $\bar{f}$, it is necessary to link $\bar{\beta}_T = (\bar{\delta}^*/\tau_w)(d\bar{p}/d\bar{x})$ to

$$G = \frac{\int_0^\delta \left(\frac{\bar{u}_e - \bar{u}}{\bar{u}_\tau}\right)^2 d\bar{y}}{\int_0^\delta \left(\frac{\bar{u}_e - \bar{u}}{\bar{u}_\tau}\right) d\bar{y}} \quad \text{(Clauser's[19] parameter)}$$

and here we have modified Alber's formulation to agree with the numerical values predicted for equilibrium flows by Mellor and Gibson[20] for which the following form was chosen

$$\bar{\beta}_T = -0.852 - 0.0596\bar{G} + (\bar{G}/5.90)^2$$

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